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A weaker criterion of asymptotic stability☆

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ABSTRACT

The classical criterion of asymptotic stability for differential equations requires the existence of a Liapunov function V with negative definite $\mathrm{d}V/\mathrm{d}t$. Successive efforts have been made to weaken the negative definiteness of $\mathrm{d}V/\mathrm{d}t$ to semi-negative definiteness. Recently, it was given an interesting result that, under the boundedness of $\mathrm{d}^{m+1}V/\mathrm{d}t^{m+1}$, the negative definiteness can be weakened to that $\mathrm{d}V/\mathrm{d}t \leq 0$ together with that $-(|\mathrm{d}V/\mathrm{d}t| + |\mathrm{d}^2V/\mathrm{d}t^2| + \cdots + |\mathrm{d}^mV/\mathrm{d}t^m| + |\mathrm{d}^{m+p}V/\mathrm{d}t^{m+p}|)$ is negative definite. Unfortunately, its basic lemma is proved to be false by a counter example and cannot support this interesting result. In this paper we re-establish the weak criterion for asymptotic stability with less requirements.

1. Introduction

Consider the nonautonomous differential system

$$\mathrm{d}x/\mathrm{d}t = f(t,x), \tag{1.1}$$

where $x \in B_H := \{x \in \mathbb{R}^n : \|x\| \leq H\}$ and $f : [0,+\infty) \times B_H \to \mathbb{R}^n$ is smooth enough to ensure existence and uniqueness of solutions of the Cauchy problem of system (1.1) and satisfies $f(t,0) \equiv 0$ for all $t \geq 0$. As shown in [1–3], the classical criterion of asymptotic stability of the zero solution of system (1.1) is that there exists a positive definite function $V(t,x)$ such that $\mathrm{d}V/\mathrm{d}t$ restricted to orbits of (1.1) is *negative definite*. In applications one sometimes is able to construct a positive definite V with nonpositive $\mathrm{d}V/\mathrm{d}t$ on orbits of (1.1) but has no way of ensuring the negative definiteness of $\mathrm{d}V/\mathrm{d}t$. In such a situation a criterion of asymptotic stability was given by Barbashin and Krasovski [4,5] for autonomous cases and periodic cases. Those results can also be found in [2]. In 1997, Barbashin and Krasovski's criterion was generalized by Ignatyev [6] for almost periodic cases. Later, more progress in improving the criterion of asymptotic stability was also made for various forms of systems [7–12]. One of those interesting results is the weaker criterion given in [12, Theorem 1], which reads that if there is a positive definite C^{m+p} function V with infinitesimal upper bound such that $\mathrm{d}^{m+1}V/\mathrm{d}t^{m+1}$ is bounded then the condition of negative definite $\mathrm{d}V/\mathrm{d}t$ can be weakened and replaced by that $\mathrm{d}V/\mathrm{d}t \leq 0$ together with that $-(|\mathrm{d}V/\mathrm{d}t| + |\mathrm{d}^2V/\mathrm{d}t^2| + \cdots + |\mathrm{d}^mV/\mathrm{d}t^m| + |\mathrm{d}^{m+p}V/\mathrm{d}t^{m+p}|)$ is negative definite (restricted to orbits of (1.1)).

The idea and efforts in [12] is very good, but some subtle procedures are involved in the analysis of convergence, which may cause confusion in understanding and inference. A counter example shows that Lemma 1 in [12], which is basic in proving his latter lemmas and his Theorem 1, is false. Thus it cannot support his result of a weaker criterion for asymptotic stability. It is also worth mentioning that Example 1 in [12] also contains a flaw so as not to demonstrate his result.

Appreciating this interesting result, in this paper we re-establish the weaker criterion for asymptotic stability with less requirements, for instance, lower smoothness and no infinitesimal upper bound.

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2. Counter example

Lemma 1 in [12] says: *Consider a C^m function $f : [0, \infty) \to \mathbb{R}$, if the limit $\lim_{x\to+\infty} f(x)$ exists, then for every sequence (y_k) with $y_k \to +\infty$ as $k \to \infty$ there exists a sequence (x_k) with $x_k - y_k \to 0$ as $k \to \infty$ such that*

$$\lim_{k\to\infty} f^{(r)}(x_k) = 0, \quad 1 \le r \le m. \tag{2.2}$$

For a counter example, consider a C^∞ function $f(x) = \cos(x^2)/x$, $x \in (0, \infty)$. It is easy to see that $\lim_{x\to+\infty} f(x) = 0$. Noting that

$$f'(x) = -2\sin(x^2) - \frac{1}{x^2}\cos(x^2), \tag{2.3}$$

$$f''(x) = -4x\cos(x^2) + \frac{2}{x}\sin(x^2) + \frac{2}{x^3}\cos(x^2), \tag{2.4}$$

we see that the sequence $y_n := (n\pi)^{1/2}$, $n = 1, 2, \ldots$, satisfies that $y_n \to +\infty$ as $n \to \infty$ and $\lim_{n\to\infty} f'(y_n) = -\lim_{n\to\infty}\cos(n\pi)/n\pi = 0$, but $\lim_{n\to\infty} f''(y_n) = \{-4(n\pi)^{1/2} + 2/(n\pi)^{3/2}\}\cos(n\pi)$ is divergent. We claim that every sequence (x_n) satisfying $x_n - y_n \to 0$ as $n \to \infty$ contains a subsequence (x_{n_k}) such that either $f'(x_{n_k})$ or $f''(x_{n_k})$ does not tend to 0, a contradiction to (2.2).

For a reductio ad absurdum, assume that both

$$\lim_{n\to\infty} f'(x_n) = 0, \qquad \lim_{n\to\infty} f''(x_n) = 0. \tag{2.5}$$

It is clear that $x_n \to +\infty$ as $n \to \infty$. For an arbitrary small $0 < \varepsilon < 1/4$, there is an integer $N > 0$ such that

$$x_n > 1/\varepsilon^{1/2}, \quad \forall n > N. \tag{2.6}$$

From the first limit in (2.5) we can choose N large enough such that $|f'(x_n)| < \varepsilon$ for all $n > N$. It follows from (2.3) that

$$|\sin(x_n^2)| < \frac{1}{2}\left(\varepsilon + \left|\frac{1}{x_n^2}\cos(x_n^2)\right|\right) < \varepsilon. \tag{2.7}$$

Thus from (2.4),

$$\begin{aligned} |f''(x_n)| &\ge |4x_n\cos(x_n^2)| - \left|\frac{2}{x_n}\sin(x_n^2)\right| - \left|\frac{2}{x_n^3}\cos(x_n^2)\right| \\ &\ge 8[1 - \sin^2(x_n^2)] - 2\varepsilon^{3/2} - \frac{2}{|x_n|^3} \\ &> 8(1 - \varepsilon^2) - 1 - \varepsilon > 1, \end{aligned} \tag{2.8}$$

where (2.6) and (2.7) are used. Clearly, (2.8) contradicts the second limit in (2.5). The claimed assertion implies that *Lemma 1 in [12] does not hold for $m \ge 2$.*

3. Corrected lemmas

Although Lemma 1 in [12] does not hold for $m \ge 2$, as indicated above, we can prove it rigorously for $m = 1$.

Lemma 1. *If a C^1 function $f : [0, \infty) \to \mathbb{R}$ satisfies that the limit $\lim_{x\to+\infty} f(x)$ exists, then for every sequence (y_k) with $y_k \to +\infty$ as $k \to \infty$ there exists a sequence (x_k) with $x_k - y_k \to 0$ as $k \to \infty$ such that $\lim_{k\to\infty} f'(x_k) = 0$.*

Proof. We first claim that for arbitrary constants $a > 0$ and $\delta > 0$ there exists a large $X = X(a, \delta) > 0$ such that for every $\tilde{y} > X$ there is $\tilde{x} \in [\tilde{y}, \tilde{y} + \delta]$ such that

$$|f'(\tilde{x})| < a. \tag{3.9}$$

Assume that this claim does not hold, i.e., there exist $\tilde{a} > 0$ and $\tilde{\delta} > 0$ such that for arbitrary $X' > 0$ there exists a $\tilde{y}_1 > X'$ which satisfies that

$$|f'(x)| \ge \tilde{a}, \quad \forall x \in [\tilde{y}_1, \tilde{y}_1 + \tilde{\delta}]. \tag{3.10}$$

By the continuity of f', it implies that either $f'(x) \ge \tilde{a}$ for all $x \in [\tilde{y}_1, \tilde{y}_1 + \tilde{\delta}]$ or $f'(x) \le -\tilde{a}$ for all $x \in [\tilde{y}_1, \tilde{y}_1 + \tilde{\delta}]$. Without loss of generality, we only consider the first case. Thus we can find a sequence $(\tilde{y}_k)$ such that $\tilde{y}_{k+1} > \tilde{y}_k + 2\tilde{\delta}$ and $f'(x) \ge \tilde{a} \forall x \in [\tilde{y}_k, \tilde{y}_k + \tilde{\delta}]$. It follows that

$$f(\tilde{y}_k + \tilde{\delta}/2) - f(\tilde{y}_k + \tilde{\delta}/4) = \int_{\tilde{y}_k+\tilde{\delta}/4}^{\tilde{y}_k+\tilde{\delta}/2} f'(\xi)\mathrm{d}\xi \ge \tilde{a}\tilde{\delta}/4,$$

a contradiction to the existence of the limit $\lim_{x\to+\infty} f(x)$. Therefore, the claimed result of (3.9) is proved.

Consider $a_\ell = 1/2^\ell$ and $\delta_\ell = 1/2^\ell$, $\ell = 1, 2, \ldots$. By the claimed result of (3.9), for each ℓ, there exists a sufficiently large $X_\ell = X(a_\ell, \delta_\ell) > 0$ such that for each $\tilde{y} > X_\ell$ there exists an $\tilde{x} := \tilde{x}(a_\ell, \delta_\ell, \tilde{y}) \in [\tilde{y}, \tilde{y} + \delta_\ell]$ which satisfies that

$$|f'(\tilde{x})| < 1/2^\ell. \tag{3.11}$$

Obviously,

$$|\tilde{x} - \tilde{y}| < 1/2^\ell. \tag{3.12}$$

Consider an arbitrary sequence (y_k) with the limit $y_k \to +\infty$ as $k \to \infty$. Corresponding to each X_ℓ, shown as above, there exists an integer $K_\ell > 0$ such that $y_k > X_\ell$ for all $k > K_\ell$. Clearly, we can choose the sequence of (K_ℓ) appropriately such that $K_{\ell+1} > K_\ell$. Now we are ready to construct a sequence (x_k) as required in the Proposition:

Step1 When $1 \le k < K_1$, let $x_k = y_k$.
Step2 When $K_1 \le k < K_2$, let $x_k = \tilde{x}(1/2, 1/2, y_k)$. By (3.11) and (3.12),

$$|x_k - y_k| < 1/2 \quad \text{and} \quad |f'(x_k)| < 1/2, \quad \forall K_1 \le k < K_2.$$

Step3 When $K_\ell \le k < K_{\ell+1}$ for some $\ell = 2, 3, \ldots$, let $x_k = \tilde{x}(1/2^\ell, 1/2^\ell, y_k)$. By (3.11) and (3.12),

$$|x_k - y_k| < 1/2^\ell \quad \text{and} \quad |f'(x_k)| < 1/2^\ell, \forall K_\ell \le k < K_{\ell+1}.$$

In this manner a sequence (x_k) is constructed, which satisfies that $x_k - y_k \to 0$ as $k \to \infty$ and that $\lim_{k\to\infty} f'(x_k) = 0$. This completes the proof. □

Our proof to this Lemma 1 exhibits clearly an idea to prove a general result for C^m $(m \ge 2)$ functions. Some steps in the proof can be employed in the proof for C^m $(m \ge 2)$ functions. The following lemma looks similar to Lemma 1 in [12] but a significant difference is that our sequence $(x_k^{(r)})$ depends on r.

Lemma 2. *If a C^m $(m \ge 2)$ function $f : [0, \infty) \to \mathbb{R}$ satisfies that the limit $\lim_{x\to+\infty} f(x)$ exists, then for every sequence (y_k) with $y_k \to +\infty$ as $k \to \infty$ and for each fixed r $(1 \le r \le m)$ there exists a sequence $(x_k^{(r)})$ with $x_k^{(r)} - y_k \to 0$ as $k \to \infty$ such that*

$$\lim_{k\to\infty} f^{(r)}(x_k^{(r)}) = 0. \tag{3.13}$$

Proof. We first claim that for given r $(1 \le r \le m)$ and for arbitrary constants $a > 0$ and $\delta > 0$ there exists a sufficiently large $X = X(r, a, \delta) > 0$ such that for every $\tilde{y} > X$ there is $\tilde{x}^{(r)} \in [\tilde{y}, \tilde{y} + \delta]$ such that

$$|f^{(r)}(\tilde{x}^{(r)})| < a. \tag{3.14}$$

The case of $r = 1$ is proved by (3.9) in the proof of Lemma 1. Suppose that this claim holds in the case of $r - 1(2 \le r \le m)$ but is not true in the case of r. Then there exist constants $\tilde{a} > 0, \tilde{\delta} > 0$ such that for arbitrary $X' > 0$ there exists a $\tilde{y}_1^{(r)} > X'$ which satisfies that $|f^{(r)}(x)| \ge \tilde{a} \forall x \in [\tilde{y}_1^{(r)}, \tilde{y}_1^{(r)} + \tilde{\delta}]$. Similarly to the statement below (3.10), there exists a sequence $(\tilde{y}_k^{(r)})$ such that $\tilde{y}_{k+1}^{(r)} > \tilde{y}_k^{(r)} + 2\tilde{\delta}$ and, without loss of generality, $f^{(r)}(x) \ge \tilde{a} \forall x \in [\tilde{y}_k^{(r)}, \tilde{y}_k^{(r)} + \tilde{\delta}]$. It follows that

$$\begin{aligned} f^{(r-1)}(\tilde{y}_k^{(r)} + t) &= f^{(r-1)}\left(\tilde{y}_k^{(r)} + \frac{\tilde{\delta}}{2}\right) - \int_{\tilde{y}_k^{(r)}+t}^{\tilde{y}_k^{(r)}+\tilde{\delta}/2} f^{(r)}(\xi)\mathrm{d}\xi \\ &\le f^{(r-1)}\left(\tilde{y}_k^{(r)} + \frac{\tilde{\delta}}{2}\right) - \tilde{a}\left(\frac{\tilde{\delta}}{2} - t\right), \quad t \in \left[0, \frac{\tilde{\delta}}{2}\right], \end{aligned} \tag{3.15}$$

$$\begin{aligned} f^{(r-1)}(\tilde{y}_k^{(r)} + t) &= f^{(r-1)}\left(\tilde{y}_k^{(r)} + \frac{\tilde{\delta}}{2}\right) + \int_{\tilde{y}_k^{(r)}+\tilde{\delta}/2}^{\tilde{y}_k^{(r)}+t} f^{(r)}(\xi)\mathrm{d}\xi \\ &\ge f^{(r-1)}\left(\tilde{y}_k^{(r)} + \frac{\tilde{\delta}}{2}\right) + \tilde{a}\left(t - \frac{\tilde{\delta}}{2}\right), \quad t \in \left[\frac{\tilde{\delta}}{2}, \tilde{\delta}\right]. \end{aligned} \tag{3.16}$$

Inequalities (3.15) and (3.16) imply respectively that

$$f^{(r-1)}(\tilde{y}_k^{(r)} + t) \le -\tilde{a}\tilde{\delta}/4, \quad \forall t \in [0, \tilde{\delta}/4], \tag{3.17}$$

if $f^{(r-1)}(\tilde{y}_k^{(r)} + \tilde{\delta}/2) \le 0$ and that

$$f^{(r-1)}(\tilde{y}_k^{(r)} + t) \ge \tilde{a}\tilde{\delta}/4, \quad \forall t \in [3\tilde{\delta}/4, \tilde{\delta}], \tag{3.18}$$

if $f^{(r-1)}(\tilde{y}_k^{(r)} + \tilde{\delta}/2) \geq 0$. Since $\tilde{y}_k^{(r)} \to +\infty$ as $t \to \infty$, for arbitrary large $X' > 0$ there is an integer $N' > 0$ such that $\tilde{y}_k^{(r)} > X'$ for all $k > N'$. Obviously, we also have $\tilde{y}_k^{(r)} + 3\tilde{\delta}/4 > X'$. Thus, in the case that $f^{(r-1)}(\tilde{y}_k^{(r)} + \tilde{\delta}/2) \leq 0$ we choose $\tilde{z}_k^{(r)} := \tilde{y}_k^{(r)}$. From (3.17) we get

$$|f^{(r-1)}(x)| \geq \tilde{a}\tilde{\delta}/4, \quad \forall x \in [\tilde{z}_k^{(r)}, \tilde{z}_k^{(r)} + \tilde{\delta}/4]. \tag{3.19}$$

In the other case, i.e., $f^{(r-1)}(\tilde{y}_k^{(r)} + \tilde{\delta}/2) > 0$, we choose $\tilde{z}_k^{(r)} := \tilde{y}_k^{(r)} + 3\tilde{\delta}/4$. From (3.18) we get the same as in (3.19). This contradicts the inductive assumption for (3.14) in the case of $r - 1$. Therefore, the claimed (3.14) is proved.

Similarly to the second part of the proof of Lemma 1, for every sequence (y_k) with $y_k \to +\infty$ as $k \to \infty$ and for each r $(2 \leq r \leq m)$ we can construct a sequence $(x_k^{(r)})$ with $x_k^{(r)} - y_k \to 0$ as $k \to \infty$ such that $\lim_{k\to\infty} f^{(r)}(x_k^{(r)}) = 0$. Thus (3.13) is proved and the proof is completed. □

By Lemma 2 we give the following result.

Lemma 3. *If a C^{m+1} $(m \geq 1)$ function $f : [0, \infty) \to \mathbb{R}$ satisfies*
(i) $f^{(m)}(x)$ is uniformly continuous on $[0, \infty)$, and
(ii) the limit $\lim_{x\to+\infty} f(x)$ exists,
then there exists a sequence (t_k) with $t_k \to +\infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \left\{|f'(t_k)| + |f''(t_k)| + \cdots + |f^{(m)}(t_k)| + |f^{(m+1)}(t_k)|\right\} = 0. \tag{3.20}$$

Proof. We first claim

$$\lim_{x\to+\infty} f^{(m)}(x) = 0. \tag{3.21}$$

Suppose that this is not true. Then there exist a constant $c > 0$ and a strictly increasing sequence (y_k^*) such that $y_k^* \to +\infty$ as $k \to \infty$ and $|f^{(m)}(y_k^*)| > c$, $k = 1, 2, \ldots$. By condition (i), there exists a constant $\delta > 0$ such that $|f^{(m)}(y_k^*) - f^{(m)}(x)| < c/2$ if $|x - y_k^*| < \delta$ for any $k = 1, 2, \ldots$, implying that $|f^{(m)}(x)| > c/2$ for all $x \in [y_k^* - \delta/2, y_k^* + \delta/2]$ and all $k = 1, 2, \ldots$. This is a contradiction to (3.14), given in the proof of Lemma 2, by condition (ii). The claimed result of (3.21) is proved.

The result of (3.21) implies the boundedness of $f^{(m)}(x)$ on $[0, \infty)$. It follows that $f^{(m-1)}(x)$ is uniformly continuous on $[0, \infty)$. By condition (ii) we similarly know that $\lim_{x\to+\infty} f^{(m-1)}(x) = 0$, as proved for (3.21). Then we see recursively that

$$\lim_{x\to+\infty} f^{(r)}(x) = 0, \quad 1 \leq r \leq m. \tag{3.22}$$

On the other hand, by Lemma 2, there exists a sequence (t_k) with $t_k \to +\infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} f^{(m+1)}(t_k) = 0. \tag{3.23}$$

From (3.22) and (3.23) we easily get (3.20). □

If we require higher smoothness, supposing $f \in C^{m+p}$ $(p > 1)$, as in [12], under the same conditions (i) and (ii) as in Lemma 3, we conclude by Lemma 1 that for each $1 < r \leq p$ there exists a sequence $(t_k^{(r)})$ with $t_k^{(r)} \to +\infty$ as $k \to \infty$ such that $\lim_{k\to\infty} f^{(m+r)}(t_k^{(r)}) = 0$. Similarly to the proof of Lemma 3 we can get

$$\lim_{k\to\infty} \left\{|f'(t_k^{(r)})| + |f''(t_k^{(r)})| + \cdots + |f^{(m)}(t_k^{(r)})| + |f^{(m+r)}(t_k^{(r)})|\right\} = 0 \tag{3.24}$$

for each $1 < r \leq p$. For $r = p$ it implies Lemma 2 in [12].

4. Weak criterion

Having those lemmas in Section 3, we can give the following weak criterion for asymptotic stability. As indicated in [2,9], *Hahn's function* is a class of continuous and monotonically increasing functions $\phi : [0, +\infty) \to [0, +\infty)$ which satisfy that $\phi(0) = 0$.

Theorem 1. *Consider differential system (1.1), where $x \in B_H$ and $f : [0, +\infty) \times B_H \to \mathbb{R}^n$ is C^m $(m \geq 1)$ and satisfies $f(t, 0) \equiv 0$ for all $t \geq 0$. If there exists a C^{m+1} function $V(t, x) : [0, +\infty) \times B_H \to \mathbb{R}$ such that the following conditions are all fulfilled on the set $[0, +\infty) \times B_H$:*

(i) *$V(t, 0) \equiv 0$ and $\phi(\|x\|) \leq V(t, x)$ for a Hahn's function ϕ,*
(ii) *$\mathrm{d}V/\mathrm{d}t \leq 0$, where $\mathrm{d}V/\mathrm{d}t = \partial V/\partial t + \partial V/\partial x \cdot f(t, x)$,*

(**iii**) $\mathrm{d}^{m+1}V/\mathrm{d}t^{m+1}$ *is bounded on the set* $[0,+\infty)\times B_H$, *where*

$$\frac{\mathrm{d}^{r+1}V}{\mathrm{d}t^{r+1}}=\frac{\partial(\mathrm{d}^rV/\mathrm{d}t^r)}{\partial t}+\frac{\partial(\mathrm{d}^rV/\mathrm{d}t^r)}{\partial x}\cdot f(t,x),\quad r=1,2,\ldots,m,$$

(**iv**) $U(t,x):=-(|\mathrm{d}V/\mathrm{d}t|+|\mathrm{d}^2V/\mathrm{d}t^2|+\cdots+|\mathrm{d}^{m+1}V/\mathrm{d}t^{m+1}|)\le-\psi(\|x\|)$ *for a Hahn's function* ψ.

then the solution $x=0$ *of differential system* (1.1) *is asymptotically stable.*

Proof. From conditions **(i)** and **(ii)** it follows that the zero solution $x=0$ of system (1.1) is stable by Theorem X.3.1 in [1] or by Theorem II.6.2 in [3]. Therefore for any $t_0>0$ and $h\in(0,H)$ there exists $\delta=\delta(t_0,h)>0$ such that a solution $x(t)=x(t,t_0,x_0)$ of system (1.1) satisfies $\|x(t)\|<h$ for all $t>t_0$ if $\|x(t_0)\|<\delta$. In the sequel, we claim that every solution $x(t)$ with $\|x(t_0)\|<\delta$ satisfies $\lim_{t\to+\infty}x(t)=0$.

Let $v(t):=V(t,x(t))$. By condition **(i)** we see that $v(t)\ge0$. From condition **(ii)** we know that $v(t)$ is monotonically nonincreasing. Hence the limit $\lim_{t\to+\infty}v(t)$ exists. From condition **(iii)** we know that $v^{(m+1)}(t)$ is bounded, implying that $v^{(m)}(t)$ is uniformly continuous. By Lemma 3, there exists a sequence (t_k) with $t_k\to+\infty$ as $k\to\infty$ such that

$$\lim_{k\to\infty}\{|v'(t_k)|+|v''(t_k)|+\cdots+|v^{(m)}(t_k)|+|v^{(m+1)}(t_k)|\}=0. \tag{4.25}$$

Note that $U(t,x(t))=-\{|v'(t)|+|v''(t)|+\cdots+|v^{(m)}(t)|+|v^{(m+1)}(t)|\}$. It follows from (4.25) that $\lim_{k\to\infty}U(t_k,x(t_k))=0$, implying by condition **(iv)** that

$$\lim_{k\to\infty}x(t_k)=0, \tag{4.26}$$

Now we claim that

$$\lim_{t\to+\infty}x(t)=0. \tag{4.27}$$

For an absurdity we assume that there exist a constant $0<c<h$ and a sequence (τ_ℓ) with $\tau_\ell\to+\infty$ as $\ell\to\infty$ such that $\|x(\tau_\ell)\|\ge c\,\forall\,\ell=1,2,\ldots$. Then, by condition **(i)**,

$$\nu:=\inf_\ell V(\tau_\ell,x(\tau_\ell))\ge\inf_\ell\phi(\|x(\tau_\ell)\|)>0. \tag{4.28}$$

On the other hand, by (4.26) there is an integer k' such that $V(t_{k'},x(t_{k'}))<\nu/2$ since V is continuous and $V(t,0)\equiv0$. Thus, by condition **(ii)**, $V(t,x(t))<\nu/2$ for all $t>t_{k'}$. Clearly, $\tau_\ell>t_{k'}$ for sufficiently large ℓ, implying that $V(\tau_\ell,x(\tau_\ell))<\nu/2$, which contradicts the definition of ν in (4.28). Therefore, (4.27) is proved and we see that the zero solution of system (1.1) is asymptotically stable. □

Corresponding to (3.24), we can obtain the following result with higher smoothness.

Corollary 1. *Consider differential system* (1.1)*, where* $x\in B_H:=\{x\in\mathbb{R}^n:\|x\|\le H\}$ *and* $f:[0,+\infty)\times B_H\to\mathbb{R}^n$ *is* C^{m+p-1} *(*$p\ge1$*) and satisfies* $f(t,0)\equiv0$ *for all* $t\ge0$*. If there exists a* C^{m+p} *function* $V(t,x):[0,+\infty)\times B_H\to\mathbb{R}$ *such that all conditions in* Theorem 1 *are satisfied but condition (iv) is replaced by*

(**iv**)′ $U(t,x):=-(|\mathrm{d}V/\mathrm{d}t|+|\mathrm{d}^2V/\mathrm{d}t^2|+\cdots+|\mathrm{d}^mV/\mathrm{d}t^m|+|\mathrm{d}^{m+r}V/\mathrm{d}t^{m+r}|)\le-\psi(\|x\|)$, $1<r\le p$*, for a Hahn's function* ψ,

then the zero solution of system (1.1) *is asymptotically stable.*

The proof follows the proof of Theorem 1 similarly, where we only need to change (4.25) into $\lim_{k\to\infty}\{|v'(t_k^{(r)})|+|v''(t_k^{(r)})|+\cdots+|v^{(m)}(t_k^{(r)})|+|v^{(m+r)}(t_k^{(r)})|\}=0$ and we omit its details. Unlike Theorem 1 in [12], our $V(t,x)$ does not need an infinitely small upper bound, so Theorem 1 in [12] is a special case of our Corollary 1.

5. Example

In Example 1 of [12] the system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\sin y-h(t)x,\qquad\frac{\mathrm{d}y}{\mathrm{d}t}=x,$$

where $h(t):=2+\sin(\frac{\pi}{2}\sqrt{t})-\sin(\frac{\pi}{2}t)$, was considered to demonstrate Theorem 1 in [12]. The author of [12] asserts that there exists a Hahn's function $c\in K$ such that $-(|\mathrm{d}V/\mathrm{d}t|+|\mathrm{d}^2V/\mathrm{d}t^2|+|\mathrm{d}^3V/\mathrm{d}t^3|+|\mathrm{d}^4V/\mathrm{d}t^4|)\le-c(x^2+y^2)$. However, when $x=0$ and $t=(4n-1)^2, n=1,2,\cdots$, one can check that $h(t)=h'(t)=0$ and therefore $-(|\mathrm{d}V/\mathrm{d}t|+|\mathrm{d}^2V/\mathrm{d}t^2|+|\mathrm{d}^3V/\mathrm{d}t^3|+|\mathrm{d}^4V/\mathrm{d}t^4|)=0\;\forall y\ne0$. This flaw implies that his Theorem 1 fails to be demonstrated by his example.

Now we consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - (1-\cos t)h(t)x^3, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - (1-\cos t)h(t)y^3, \tag{5.29}$$

where $h(t) := (\tanh t + \operatorname{sech} t)/2$. Clearly, the function $V(t,x,y) = (x^2+y^2)/2$ satisfies that $V(t,0,0) \equiv 0$ and $V(t,x,y) > 0$ for all $(x,y) \neq (0,0)$. Moreover, $\mathrm{d}V/\mathrm{d}t = -(1-\cos t)h(t)(x^4+y^4) \leq 0$, implying that the zero solution of system (5.29) is stable. One can calculate

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2} = -h\sin t(x^4+y^4) - (1-\cos t)\mathcal{L}(t,x,y),$$
$$\frac{\mathrm{d}^3V}{\mathrm{d}t^3} = -h\cos t(x^4+y^4) - 2\sin t\,\mathcal{L}(t,x,y) - (1-\cos t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t,x,y),$$

where $\mathcal{L}(t,x,y) := 4xyh(x^2-y^2) - 4h(1-\cos t)(x^6+y^6) + h'(x^4+y^4)$. For $(t,x,y) \in [0,+\infty) \times B_H$, where $H < 1$ is a sufficiently small positive number,

$$\begin{aligned}|\mathcal{L}(t,x,y)| &\leq |4xyh(x^2-y^2)| + |4h(1-\cos t)(x^6+y^6)| + |h'(x^4+y^4)| \\ &\leq 2(x^2+y^2)|x^2-y^2| + 10(x^4+y^4) \\ &\leq 12(x^4+y^4).\end{aligned}$$

Similarly, we get $|\mathrm{d}^3V/\mathrm{d}t^3| < 800(x^4+y^4)$, which means that $\mathrm{d}^3V/\mathrm{d}t^3$ is bounded on the set $[0,+\infty)\times B_H$, as required by condition **(iii)** in Theorem 1.

Let ε sufficiently small positive number ($\varepsilon = 10^{-3}$). It is easy to see that

$$\begin{aligned}\left|\frac{\mathrm{d}^3V}{\mathrm{d}t^3}\right| &= |-h\cos t(x^4+y^4) - 2\sin t\,\mathcal{L}(t,x,y) - (1-\cos t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t,x,y)| \\ &\geq |h\cos t(x^4+y^4)| - 2|\sin t\,\mathcal{L}(t,x,y)| - |(1-\cos t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t,x,y)| \\ &\geq \frac{1}{4}(x^4+y^4) \geq \frac{1}{8}(x^2+y^2)^2\end{aligned}$$

for all $t \in [2k\pi - \varepsilon, 2k\pi + \varepsilon]$, $k = 0, 1, \ldots$. On the other hand,

$$\left|\frac{\mathrm{d}V}{\mathrm{d}t}\right| = (1-\cos t)h(x^4+y^4) > \left(\frac{1-\cos\varepsilon}{4}\right)(x^4+y^4) \geq \left(\frac{1-\cos\varepsilon}{8}\right)(x^2+y^2)^2$$

for all $t \in (2k\pi + \varepsilon, 2(k+1)\pi - \varepsilon)$, $k = 0, 1, \ldots$. So, for $t > 0$ we obtain

$$\left|\frac{\mathrm{d}V}{\mathrm{d}t}\right| + \left|\frac{\mathrm{d}^2V}{\mathrm{d}t^2}\right| + \left|\frac{\mathrm{d}^3V}{\mathrm{d}t^3}\right| \geq \left|\frac{\mathrm{d}^3V}{\mathrm{d}t^3}\right| + \left|\frac{\mathrm{d}V}{\mathrm{d}t}\right| \geq \left(\frac{1-\cos\varepsilon}{8}\right)(x^2+y^2)^2 = \psi((x^2+y^2)^{\frac{1}{2}}),$$

where $\psi(\xi) := (1-\cos\varepsilon)\xi^4/8$ is a Hahn's function. Thus the condition **(iv)** in our Theorem 1 is fulfilled. It implies that the zero solution of system (5.29) is asymptotically stable.

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